On the maximal monotonicity of the sum of a maximal monotone linear relation and the subdifferential operator of a sublinear function

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January 1, 2010

Abstract

The most important open problem in Monotone Operator Theory concerns the maximal monotonicity of the sum of two maximal monotone operators provided that Rockafellar's constraint qualification holds.

In this note, we provide a new maximal monotonicity result for the sum of a maximal monotone relation and the subdifferential operator of a proper, lower semicontinuous, sublinear function. The proof relies on Rockafellar's formula for the Fenchel conjugate of the sum as well as some results on the Fitzpatrick function.

2000 Mathematics Subject Classification:
Primary 47A06, 47H05; Secondary 47B65, 49N15, 52A41, 90C25

Keywords: Constraint qualification, convex function, convex set, Fenchel conjugate, Fitzpatrick function, linear relation, maximal monotone operator, multifunction, monotone operator, set-valued operator, subdifferential operator, sublinear function, Rockafellar's sum theorem.

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1 Introduction

Throughout this paper, we assume that X is a real Banach space with norm $\|\cdot\|$, that X^* is the continuous dual of X, and that X and X^* are paired by $\langle\cdot,\cdot\rangle$. Let $A\colon X \rightrightarrows X^*$ be a *set-valued operator* (also known as multifunction) from X to X^*, i.e., for every $x \in X$, $Ax \subseteq X^*$, and let $\operatorname{gra} A = \big\{(x,x^*) \in X \times X^* \mid x^* \in Ax\big\}$ be the *graph* of A. Recall that A is *monotone* if

$$(1) \qquad \big(\forall (x,x^*) \in \operatorname{gra} A\big)\big(\forall (y,y^*) \in \operatorname{gra} A\big) \quad \langle x-y, x^*-y^*\rangle \geq 0,$$

and *maximal monotone* if A is monotone and A has no proper monotone extension (in the sense of graph inclusion). We say A is a *linear relation* if $\operatorname{gra} A$ is a linear subspace. Monotone operators have proven to be a key class of objects in modern Optimization and Analysis; see, e.g., the books [8, 9, 10, 13, 18, 19, 17, 26] and the references therein. (We also adopt standard notation used in these books: $\operatorname{dom} A = \big\{x \in X \mid Ax \neq \varnothing\big\}$ is the *domain* of A. Given a subset C of X, $\operatorname{int} C$ is the *interior* of C, and $\overline{C}$ is the *closure* of C. We set $C^\perp := \{x^* \in X^* \mid (\forall c \in C)\, \langle x^*, c\rangle = 0\}$ and $S^\perp := \{x^{**} \in X^{**} \mid (\forall s \in S)\, \langle x^{**}, s\rangle = 0\}$ for a set $S \subseteq X^*$. The *indicator function* of C, written as ι_C, is defined at $x \in X$ by

$$(2) \qquad \iota_C(x) := \begin{cases} 0, & \text{if } x \in C; \\ \infty, & \text{otherwise.} \end{cases}$$

Given $f\colon X \to \,]-\infty,+\infty]$, we set $\operatorname{dom} f = f^{-1}(\mathbb{R})$ and $f^*\colon X^* \to [-\infty,+\infty] : x^* \mapsto \sup_{x\in X}(\langle x,x^*\rangle - f(x))$ is the *Fenchel conjugate* of f. If f is convex and $\operatorname{dom} f \neq \varnothing$, then $\partial f\colon X \rightrightarrows X^* \colon x \mapsto \big\{x^* \in X^* \mid (\forall y \in X)\ \langle y-x, x^*\rangle + f(x) \leq f(y)\big\}$ is the *subdifferential operator* of f. Recall that f is *sublinear* if $f(0) = 0$, $f(x+y) \leq f(x) + f(y)$, and $f(\lambda x) = \lambda f(x)$ for all $x, y \in \operatorname{dom} f$ and $\lambda > 0$. Finally, the *closed unit ball* in X is denoted by $B_X := \big\{x \in X \mid \|x\| \leq 1\big\}$.) Throughout, we shall identify X with its canonical image in the bidual space X^{**}. Furthermore, $X \times X^*$ and $(X \times X^*)^* = X^* \times X^{**}$ are likewise paired via $\langle (x,x^*),(y^*,y^{**})\rangle = \langle x,y^*\rangle + \langle x^*,y^{**}\rangle$, where $(x,x^*) \in X \times X^*$ and $(y^*,y^{**}) \in X^* \times X^{**}$.

Let A and B be maximal monotone operators from X to X^*. Clearly, the *sum operator* $A+B\colon X \rightrightarrows X^* \colon x \mapsto Ax + Bx = \big\{a^* + b^* \mid a^* \in Ax \text{ and } b^* \in Bx\big\}$ is monotone. Rockafellar's [16, Theorem 1] guarantees maximal monotonicity of $A+B$ under the classical *constraint qualification* $\operatorname{dom} A \cap \operatorname{int}\operatorname{dom} B \neq \varnothing$ when X is reflexive. The most famous open problem concerns the behaviour in nonreflexive Banach spaces. See Simons' monograph [19] for a comprehensive account of the recent developments.

Now we focus on the special case when A is a *linear relation* and B is the subdifferential operator of a *sublinear* function f. We show that the sum theorem is true in this setting. Recently, linear relations have increasingly been studied in detail; see, e.g., [1, 2, 3, 4, 5, 6, 7, 14, 21, 23, 24, 25] and Cross' book [11] for general background on linear relations.

The remainder of this paper is organized as follows. In Section 2, we collect auxiliary results for future reference and for the reader's convenience. The main result (Theorem 3.1) is proved in Section 3.

2 Auxiliary Results

Fact 2.1 (Rockafellar) (See [15, Theorem 3], [19, Corollary 10.3 and Theorem 18.1], or [26, Theorem 2.8.7(iii)].)
Let $f, g : X \to \left]-\infty, +\infty\right]$ be proper convex functions. Assume that there exists a point $x_0 \in \operatorname{dom} f \cap \operatorname{dom} g$ such that g is continuous at x_0. Then for every $z^* \in X^*$, there exists $y^* \in X^*$ such that

$$(f+g)^*(z^*) = f^*(y^*) + g^*(z^* - y^*). \tag{3}$$

Furthermore, $\partial(f+g) = \partial f + \partial g$.

Fact 2.2 (Fitzpatrick) (See [12, Corollary 3.9].) *Let $A \colon X \rightrightarrows X^*$ be maximal monotone, and set*

$$F_A \colon X \times X^* \to \left]-\infty, +\infty\right] : (x, x^*) \mapsto \sup_{(a,a^*) \in \operatorname{gra} A} \big(\langle x, a^* \rangle + \langle a, x^* \rangle - \langle a, a^* \rangle\big), \tag{4}$$

which is the Fitzpatrick function *associated with A. Then for every $(x, x^*) \in X \times X^*$, the inequality $\langle x, x^* \rangle \le F_A(x, x^*)$ is true, and equality holds if and only if $(x, x^*) \in \operatorname{gra} A$.*

Fact 2.3 (Simons) (See [19, Theorem 24.1(c)].) *Let $A, B : X \rightrightarrows X^*$ be maximal monotone operators. Assume $\bigcup_{\lambda>0} \lambda \left[P_X(\operatorname{dom} F_A) - P_X(\operatorname{dom} F_B)\right]$ is a closed subspace, where $P_X : (x, x^*) \in X \times X^* \to x$. If*

$$(x, x^*) \text{ is monotonically related to } \operatorname{gra}(A+B) \Rightarrow x \in \operatorname{dom} A \cap \operatorname{dom} B, \tag{5}$$

then $A + B$ is maximal monotone.

Fact 2.4 (Simons) (See [19, Lemma 19.7 and Section 22].) *Let $A : X \rightrightarrows X^*$ be a monotone linear relation such that $\operatorname{gra} A \neq \varnothing$. Then the function*

$$g \colon X \times X^* \to \left]-\infty, +\infty\right] : (x, x^*) \mapsto \langle x, x^* \rangle + \iota_{\operatorname{gra} A}(x, x^*) \tag{6}$$

is proper and convex.

Fact 2.5 (Simons) (See [20, Lemma 2.2].) *Let $f : X \to \left]-\infty, +\infty\right]$ be proper, lower semicontinuous, and convex. Let $x \in X$ and $\lambda \in \mathbb{R}$ be such that* $\inf f < \lambda < f(x) \leq +\infty$, *and set*

$$K := \sup_{a \in X, a \neq x} \frac{\lambda - f(a)}{\|x - a\|}.$$

Then $K \in \left]0, +\infty\right[$ and for every $\varepsilon \in \left]0, 1\right[$, there exists $(y, y^*) \in \operatorname{gra} \partial f$ such that

$$\langle y - x, y^* \rangle \leq -(1 - \varepsilon) K \|y - x\| < 0. \tag{7}$$

Fact 2.6 (See [26, Therorem 2.4.14].) *Let $f : X \to \left]-\infty, +\infty\right]$ be a sublinear function. Then the following hold.*

(i) $\partial f(x) = \{x^* \in \partial f(0) \mid \langle x^*, x \rangle = f(x)\}, \quad \forall x \in \operatorname{dom} f$.

(ii) $\partial f(0) \neq \varnothing \Leftrightarrow f$ *is lower semicontinuous at* 0.

(iii) *If f is lower semicontinuous, then* $f = \sup \langle \cdot, \partial f(0) \rangle$.

Fact 2.7 (See [13, Proposition 3.3 and Proposition 1.11].) *Let $f : X \to \left]-\infty, +\infty\right]$ be a lower semicontinuous convex and* $\operatorname{int} \operatorname{dom} f \neq \varnothing$. *Then f is continuous on* $\operatorname{int} \operatorname{dom} f$ *and* $\partial f(x) \neq \varnothing$ *for every* $x \in \operatorname{int} \operatorname{dom} f$.

Lemma 2.8 *Let $f : X \to \left]-\infty, +\infty\right]$ be a sublinear function. Then* $\operatorname{dom} f + \operatorname{int} \operatorname{dom} f = \operatorname{int} \operatorname{dom} f$.

Proof. The result is trivial when $\operatorname{int} \operatorname{dom} f = \varnothing$ so we assume that $x_0 \in \operatorname{int} \operatorname{dom} f$. Then there exists $\delta > 0$ such that $x_0 + \delta B_X \subseteq \operatorname{dom} f$. By sublinearity, $\forall y \in \operatorname{dom} f$, we have $y + x_0 + \delta B_X \subseteq \operatorname{dom} f$. Hence

$$y + x_0 \in \operatorname{int} \operatorname{dom} f.$$

Then $\operatorname{dom} f + \operatorname{int} \operatorname{dom} f \subseteq \operatorname{int} \operatorname{dom} f$. Since $0 \in \operatorname{dom} f$, $\operatorname{int} \operatorname{dom} f \subseteq \operatorname{dom} f + \operatorname{int} \operatorname{dom} f$. Hence $\operatorname{dom} f + \operatorname{int} \operatorname{dom} f = \operatorname{int} \operatorname{dom} f$. ■

Lemma 2.9 *Let $A : X \rightrightarrows X^*$ be a maximal monotone linear relation, and let $z \in X \cap (A0)^\perp$. Then $z \in \overline{\operatorname{dom} A}$.*

Proof. Suppose to the contrary that $z \notin \overline{\operatorname{dom} A}$. Then the Separation Theorem provides $w^* \in X^*$ such that

$$\langle z, w^* \rangle > 0 \quad \text{and} \quad w^* \in \overline{\operatorname{dom} A}^\perp. \tag{8}$$

Thus, $(0, w^*)$ is monotonically related to $\operatorname{gra} A$. Since A is maximal monotone, we deduce that $w^* \in A0$. By assumption, $\langle z, w^*\rangle = 0$, which contradicts (8). Hence, $z \in \overline{\operatorname{dom} A}$. ■

The proof of the next result follows closely the proof of [19, Theorem 53.1].

Lemma 2.10 *Let $A : X \rightrightarrows X^*$ be a monotone linear relation, and let $f : X \to \,]-\infty, +\infty]$ be a proper lower semicontinuous convex function. Suppose that $\operatorname{dom} A \cap \operatorname{int} \operatorname{dom} \partial f \neq \varnothing$, $(z, z^*) \in X \times X^*$ is monotonically related to $\operatorname{gra}(A + \partial f)$, and that $z \in \operatorname{dom} A$. Then $z \in \operatorname{dom} \partial f$.*

Proof. Let $c_0 \in X$ and $y^* \in X^*$ be such that

$$c_0 \in \operatorname{dom} A \cap \operatorname{int} \operatorname{dom} \partial f \quad \text{and} \quad (z, y^*) \in \operatorname{gra} A. \tag{9}$$

Take $c_0^* \in Ac_0$, and set

$$M := \max\left\{\|y^*\|, \|c_0^*\|\right\}, \tag{10}$$

$D := [c_0, z]$, and $h := f + \iota_D$. By (9), Fact 2.7 and Fact 2.1, $\partial h = \partial f + \partial \iota_D$. Set $H : X \to \,]-\infty, +\infty] : x \mapsto h(x + z) - \langle z^*, x\rangle$. It remains to show that

$$0 \in \operatorname{dom} \partial H. \tag{11}$$

If $\inf H = H(0)$, then (11) holds. Now suppose that $\inf H < H(0)$. Let $\lambda \in \mathbb{R}$ be such that $\inf H < \lambda < H(0)$, and set

$$K_\lambda := \sup_{H(x) < \lambda} \frac{\lambda - H(x)}{\|x\|}. \tag{12}$$

We claim that

$$K_\lambda \leq M.$$

By Fact 2.5, we have $K_\lambda \in \,]0, \infty[$ and $\forall \varepsilon \in \,]0, 1[$, by $\operatorname{gra} \partial H = \operatorname{gra} \partial h - (z, z^*)$ there exists $(x, x^*) \in \operatorname{gra} \partial h$ such that

$$\langle x - z, x^* - z^*\rangle \leq -(1 - \varepsilon) K_\lambda \|x - z\| < 0. \tag{13}$$

Since $\partial h = \partial f + \partial \iota_D$, there exists $t \in [0, 1]$ with $x_1^* \in \partial f(x)$ and $x_2^* \in \partial \iota_D(x)$ such that $x = tc_0 + (1 - t)z$ and $x^* = x_1^* + x_2^*$. Then $\langle x - z, x_2^*\rangle \geq 0$. Thus, by (13),

$$\langle x - z, x_1^* - z^*\rangle \leq \langle x - z, x_1^* + x_2^* - z^*\rangle \leq -(1 - \varepsilon) K_\lambda \|x - z\| < 0. \tag{14}$$

As $x = tc_0 + (1 - t)z$ and A is a linear relation, we have $(x, tc_0^* + (1 - t)y^*) \in \operatorname{gra} A$. Since (z, z^*) is monotonically related to $\operatorname{gra}(A + \partial f)$, by (10),

$$\langle x - z, x_1^* - z^*\rangle \geq -\langle x - z, tc_0^* + (1 - t)y^*\rangle \geq -M\|x - z\|. \tag{15}$$

Combining (15) and (14), we obtain

$$-M\|x-z\| \le -(1-\varepsilon)K_\lambda\|x-z\| < 0. \tag{16}$$

Hence, $(1-\varepsilon)K_\lambda \le M$. Letting $\varepsilon \downarrow 0$, we deduce that $K_\lambda \le M$. Then, by (12) and letting $\lambda \uparrow H(0)$, we get

$$H(y) + M\|y\| \ge H(0), \quad \forall y \in X. \tag{17}$$

By [19, Example 7.1], $0 \in \operatorname{dom} \partial H$. Hence (11) holds and thus $z \in \operatorname{dom} \partial f$. ■

3 Main Result

Theorem 3.1 *Let $A : X \rightrightarrows X^*$ be a maximal monotone linear relation, let $f : X \to \left]-\infty, +\infty\right]$ be a proper lower semicontinuous sublinear function, and suppose that* $\operatorname{dom} A \cap \operatorname{int} \operatorname{dom} \partial f \neq \varnothing$. *Then $A + \partial f$ is maximal monotone.*

Proof. Let $(z, z^*) \in X \times X^*$ and suppose that

$$(z, z^*) \text{ is monotonically related to } \operatorname{gra}(A + \partial f). \tag{18}$$

By Fact 2.2, $\operatorname{dom} A \subseteq P_X(F_A)$ and $\operatorname{dom} \partial f \subseteq P_X(F_{\partial f})$. Hence,

$$\bigcup_{\lambda > 0} \lambda\big(P_X(\operatorname{dom} F_A) - P_X(\operatorname{dom} F_{\partial f})\big) = X. \tag{19}$$

Thus, by Fact 2.3, it suffices to show that

$$z \in \operatorname{dom} A \cap \operatorname{dom} \partial f. \tag{20}$$

We have

$$\begin{aligned} &\langle z, z^*\rangle - \langle z, x^*\rangle - \langle x, z^*\rangle + \langle x, x^*\rangle + \langle x - z, y^*\rangle \\ &= \langle z - x, z^* - x^* - y^*\rangle \ge 0, \quad \forall (x, x^*) \in \operatorname{gra} A, (x, y^*) \in \operatorname{gra} \partial f. \end{aligned} \tag{21}$$

By Fact 2.6(ii), $\partial f(0) \neq \varnothing$. By (21),

$$\inf\left[\langle z, z^*\rangle - \langle z, A0\rangle - \langle z, \partial f(0)\rangle\right] \ge 0.$$

Thus,

$$z \in X \cap (A0)^\perp. \tag{22}$$

Then, by Fact 2.6(iii),

$$\langle z, z^* \rangle \geq f(z).$$

Thus,

$$z \in \operatorname{dom} f. \tag{23}$$

By (22) and Lemma 2.9, we have

$$z \in \overline{\operatorname{dom} A}. \tag{24}$$

By Fact 2.6(i), $y^* \in \partial f(0)$ as $y^* \in \partial f(x)$. Then $\langle x - z, y^* \rangle \leq f(x - z), \quad \forall y^* \in \partial f(x)$. Thus, by (21), we have

$$\langle z, z^* \rangle - \langle z, x^* \rangle - \langle x, z^* \rangle + \langle x, x^* \rangle + f(x - z) \geq 0, \quad \forall (x, x^*) \in \operatorname{gra} A, x \in \operatorname{dom} \partial f. \tag{25}$$

Let $C := \operatorname{int} \operatorname{dom} f$. Then by Fact 2.7, we have

$$\langle z, z^* \rangle - \langle z, x^* \rangle - \langle x, z^* \rangle + \langle x, x^* \rangle + f(x - z) \geq 0, \quad \forall (x, x^*) \in \operatorname{gra} A, x \in C. \tag{26}$$

Set $j := (f(\cdot - z) + \iota_C) \oplus \iota_{X^*}$ and

$$g \colon X \times X^* \to \left]-\infty, +\infty\right] : (x, x^*) \mapsto \langle x, x^* \rangle + \iota_{\operatorname{gra} A}(x, x^*). \tag{27}$$

By Fact 2.4, g is convex. Hence,

$$h := g + j \tag{28}$$

is convex as well. Let

$$c_0 \in \operatorname{dom} A \cap C. \tag{29}$$

By Lemma 2.8 and (23), $z + c_0 \in \operatorname{int} \operatorname{dom} f$. Then there exists $\delta > 0$ such that $z + c_0 + \delta B_X \subseteq \operatorname{dom} f$ and $c_0 + \delta B_X \subseteq \operatorname{dom} f$. By (24), $z + c_0 \in \overline{\operatorname{dom} A}$ since $\operatorname{dom} A$ is a linear subspace. Thus there exists $b \in \frac{1}{2}\delta B_X$ such that $z + c_0 + b \in \operatorname{dom} A \cap \operatorname{int} \operatorname{dom} f$. Let $v^* \in A(z + c_0 + b)$. Since $c_0 + b \in \operatorname{int} \operatorname{dom} f$,

$$(z + c_0 + b, v^*) \in \operatorname{gra} A \cap \Big(\operatorname{int} C \cap \operatorname{int} \operatorname{dom} f(\cdot - z) \times X^*\Big) = \operatorname{dom} g \cap \operatorname{int} \operatorname{dom} j \neq \varnothing. \tag{30}$$

By Fact 2.1 and Fact 2.7, there exists $(y^*, y^{**}) \in X^* \times X^{**}$ such that

$$\begin{aligned}
h^*(z^*, z) &= g^*(y^*, y^{**}) + j^*(z^* - y^*, z - y^{**}) \\
&= g^*(y^*, y^{**}) + \iota_{\{0\}}(z - y^{**}) + \sup_{x \in C} [\langle x, z^* - y^* \rangle - f(x - z)] \\
&\geq g^*(y^*, y^{**}) + \iota_{\{0\}}(z - y^{**}) + \sup_{x \in z + C} [\langle x, z^* - y^* \rangle - f(x - z)] \text{ (by Lemma 2.8 and (23))} \\
&= g^*(y^*, y^{**}) + \iota_{\{0\}}(z - y^{**}) + \langle z, z^* - y^* \rangle + \sup_{y \in C} [\langle y, z^* - y^* \rangle - f(y)] \\
&= g^*(y^*, y^{**}) + \iota_{\{0\}}(z - y^{**}) + \langle z, z^* - y^* \rangle + \sup_{\{y \in C, k > 0\}} [\langle ky, z^* - y^* \rangle - f(ky)] \\
&= g^*(y^*, y^{**}) + \iota_{\{0\}}(z - y^{**}) + \langle z, z^* - y^* \rangle + \sup_{\{y \in C, k > 0\}} k [\langle y, z^* - y^* \rangle - f(y)] \\
&\geq g^*(y^*, y^{**}) + \iota_{\{0\}}(z - y^{**}) + \langle z, z^* - y^* \rangle. \qquad (31)
\end{aligned}$$

By (26), we have, for every $(x, x^*) \in \operatorname{gra} A \cap (C \times X^*)$, $\langle (x, x^*), (z^*, z) \rangle - h(x, x^*) = \langle x, z^* \rangle + \langle z, x^* \rangle - \langle x, x^* \rangle - f(x - z) \leq \langle z, z^* \rangle$. Consequently,

$$h^*(z^*, z) \leq \langle z, z^* \rangle. \qquad (32)$$

Combining (31) with (32), we obtain

$$g^*(y^*, y^{**}) + \langle z, z^* - y^* \rangle + \iota_{\{0\}}(z - y^{**}) \leq \langle z, z^* \rangle. \qquad (33)$$

Therefore, $y^{**} = z$. Hence $g^*(y^*, z) + \langle z, z^* - y^* \rangle \leq \langle z, z^* \rangle$. Since $g^*(y^*, z) = F_A(z, y^*)$, we deduce that $F_A(z, y^*) \leq \langle z, y^* \rangle$. By Fact 2.2,

$$(z, y^*) \in \operatorname{gra} A \qquad (34)$$

Hence

$$z \in \operatorname{dom} A.$$

Apply Lemma 2.10 to obtain $z \in \operatorname{dom} \partial f$. Then $z \in \operatorname{dom} A \cap \operatorname{dom} \partial f$. Hence $A + B$ is maximal monotone. ■

Remark 3.2 Verona and Verona (see [22, Corollary 2.9(a)] or [19, Theorem 53.1]) showed the following: "Let $f : X \to]-\infty, +\infty]$ be proper, lower semicontinuous, and convex, let $A : X \rightrightarrows X^*$ be maximal monotone, and suppose that $\operatorname{dom} A = X$. Then $\partial f + A$ is maximal monotone." Note that Theorem 3.1 cannot be deduced from this result because $\operatorname{dom} A$ need not have full domain.

Acknowledgment

Heinz Bauschke was partially supported by the Natural Sciences and Engineering Research Council of Canada and by the Canada Research Chair Program. Xianfu Wang was partially supported by the Natural Sciences and Engineering Research Council of Canada.

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